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The use of fractional exponential creep kernels for long-term behavior of laterally loaded piles in permafrost

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Abstract. A theory for long-term prediction of laterally loaded piles in a hereditary-elastic medium is presented. A mixed formulation approach with the aid of calculus of variations ensures continuity of pile displacements, bending moments and soil reactions. Finally, an asymptotic estimate of the pile behavior for very long time ($t \to \infty$) is obtained using a fractional exponential operator derived from pile tests.

1. Introduction

Pile foundations in frozen soils are increasingly in demand, because in addition to above-graded heated structures and bridge abutments their use is now adopted for the construction of oil pipelines and offshore drilling platforms. Three firmly established valid approaches to predict the behavior of laterally loaded piles are: Modulus of Subgrade Reaction Method (Hetenyi [5], Nixon [10], Neukirchner and Nixon [11], Neukirchner [12]), Elastic Continuum Approach (Poulos [15]) and Pressure Deflection-Curve Method (Ladanyi [9], Rowley, Watson and Ladanyi [20, 21]), respectively. In frozen soils the first and the third method are most widely utilized. Furthermore, in frozen soils until now, the numerical solutions for laterally loaded piles in permafrost have been obtained by means of finite difference techniques applied to a coupled system of partial differential equations. Based on a Winkler model, together with a secondary creep law, Nixon [10] obtained displacements, shear stresses and bending moments along the pile length by employing classical relationships for bending of beams. In a recent study, the authors (Foriero and Ladanyi [3]) analyzed the same problem by the finite element technique. The reaction redistribution along the pile length was determined using a non-linear creep law and the concept of non-stationarity was introduced.

In this paper, a variational method is used to analyze laterally loaded piles in frozen soils with consideration of stationary creep (Foriero and Ladanyi [4]). For non-stationary creep, the use of numerical techniques is inevitable and one can consult the above mentioned study (Foriero and Ladanyi [3]).

2. Variational methods

Classical variational methods provide effective approximate solutions to physical problems (Oden and Reddy [13], Owen and Hinton [14], Reddy [19], Zienkiewics [22]). The involved formulative and computational efforts are significantly less than those of the finite difference and finite element methods. In addition, these approximate solutions are continuous functions of position (on the domain), as opposed to piecewise continuous or discretely defined functions obtained in the finite difference and finite element methods.

For a laterally loaded pile in a hereditary elastic medium, two variational principles are applicable: namely the principle of minimum potential energy (involving displacements) and the principle of minimum complementary energy (involving stresses) (Reissner [17, 18]). In the former, displacements are primary variables (essential conditions) and stresses are secondary variables (natural conditions), whereas in the complementary energy principle the reverse occurs. In the literature (Foriero and Ladanyi [3], Klein [8], Oden and Reddy [13]), both variational principles have been effective for obtaining approximate solutions of boundary value problems. However, in the approximate solutions the primary variables are continuous while the secondary variables are not.

The objective herein is to utilize the calculus of variations to obtain approximate solutions in such manner that no preference is given to either of the two kinds of essential conditions (displacement or stresses) occurring in the theory. A variational problem for laterally loaded piles in permafrost is formulated wherein displacements, moments and reactions occur as primary variables (essential conditions). This mixed type formulation was first applied by Reissner [17, 18] in elasticity and thus, by analogy, the method is valid for laterally loaded piles as well.

3. Laterally loaded piles in a hereditary-elastic medium

The differential equation for time-independent displacement $w(x)$ of a pile in an elastic medium is written as (Hetenyi [5])

$$\mathrm{D}^2(EI\mathrm{D}^2 w) + kw = q \quad (0 \leqslant x \leqslant L,\ \mathrm{D} \equiv \mathrm{d}/\mathrm{d}x)\,, \tag{3.1}$$

where k is the reaction modulus of the soil in units of stress and q is any disturbance (load) acting on the pile. This latter quantity for the problem dealt with in this article does not stand for an $\mathrm{L}_2[0, L]$-function, but merely for the linear combination of the Dirac-delta functional and its derivative

$$q = F(0)\delta(x+0) - M(0)\delta'(x+0)\,, \tag{3.2}$$

whose domain of definition is at most the space of continuously differentiable functions on $[0, L]$. If one assumes the pile has constant cross section and rigidity modulus EI independent of depth (constant EI) and is embedded in a viscoelastic medium with modulus k, then k must be replaced by its viscoelastic modulus operator $k(I - H)$, where

$$(If)(t) \equiv f(t), \quad (Hf)(t) \equiv (h * f)(t) = \int_{\tau=0}^{t} h(t-\tau) f(\tau)\,\mathrm{d}\tau\,. \tag{3.3}$$

Thus equation (3.1) for time-dependent displacement $w(x, t)$ is rewritten as

$$EI(\mathrm{D}_1^4 w) + k(I - H)w = q \quad (0 \leqslant x \leqslant L,\ \mathrm{D}_1 \equiv \partial/\partial x)\,, \tag{3.4}$$

which is the bending equation of a pile embedded in a hereditary elastic medium satisfying the fading memory and closed cycle conditions (Rabotnov [16]). Inherent to expression (3.4) is the fact that the pile displacement at any level is related to global reaction via

$$w = -k^{-1}(I + C)p \quad \text{or} \quad p = -k(I - H)w\,, \tag{3.5}$$

where C is a convolution operator with kernel $c(t)$ (Doetsch [2]). The first relation in (3.5) is explicitly

$$w(x, t) = -k^{-1}p(x, t) - k^{-1}\int_{\tau=0}^{t} c(t-\tau)p(x, \tau)\,\mathrm{d}\tau \tag{3.6}$$

and in terms of convolution becomes

$$w(x, t) = -k^{-1}p(x, t) - k^{-1}(c * p(x, \cdot))(t)\,. \tag{3.7}$$

The analysis begins by defining a sign convention for displacements, reactions and moments (Fig. 1). A Reissner [17, 18] type functional is obtained thereafter by considering energy principles and is verified by taking the variation of the respective essential variables (i.e. the result yields the original integral equation (3.4)), where the variation with respect to p of the convolution integral of $p(x, t)$ with respect to time t is excluded because of the fading memory condition. The problem therefore deals with solving for that specific function $w(x, t)$ which yields

$$\begin{aligned}\min\Big\{J(w, p, M) \equiv \int_{x=0}^{L} \Big[& -M(x, t)(\mathrm{D}_1^2 w) + p(x, t)w(x, t) + (2EI)^{-1}(M(x, t))^2 \\ & + (2k)^{-1}(p(x, t))^2 + k^{-1}p(x, t)\int_{\tau=0}^{t} c(t-\tau)p(x, \tau)\,\mathrm{d}\tau \\ & + F(0)\delta(x+0)w(x, t) - M(0)\delta(x+0)(\mathrm{D}_1 w)(x, t)\Big]\,\mathrm{d}x:\end{aligned}$$

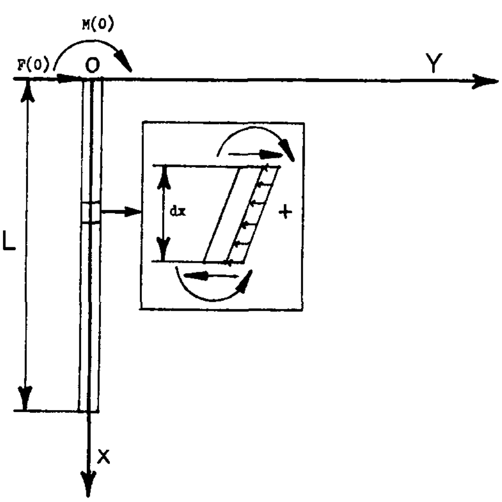

Fig. 1. Sign convention for laterally loaded piles.

$$p(x, t) = -kw(x, t) - k \int_{\tau=0}^{t} h(t-\tau) w(x, \tau)\, d\tau ,$$

$$M(x, t) = EI(D_1^2 w)(x, t) ,$$

$$\left.(D_1^2 w)(0+, t) = (D_1^3 w)(0+, t) = (D_1^2 w)(L-, t) = (D_1^3 w)(L-, t) = 0\right\} , \tag{3.8}$$

where $w(x, t)$, $M(x, t)$ and $p(x, t)$ are the essential or primary variables and the action of $\delta(x + 0)$ (modified Dirac delta functional) is defined by

$$\int_{x=0}^{\infty} \delta(x+0) f(x)\, dx = f(0+) = f(0) \quad (f \in C[0, a],\ a > 0) . \tag{3.9}$$

This modification of the Dirac delta functional is in consequence of the fact that forces and moments are applied infinitesimally close to the upper boundary of the pile (Fig. 1). It is assumed that no forces and moments act on the lower boundary – i.e. the pile is very long.

The variational method is now applied as follows to Reissner functional (3.8), which functional incorporates the modified Dirac delta function as well as its derivative. A complete orthonormal family of functions derived from a self-adjoint homogeneous boundary value problem (Kamke [7, Chap. VIII]) must be constructed such that the homogeneous boundary conditions are compatible with the quantities $w(0+, t)$ and $(D_1 w)(0+, t)$ arising from the modified Dirac delta functional and its derivative respectively, as well as the homogeneous boundary conditions $EI(D_1^2 w)(L-, t) = 0$ and $EI(D_1^3 w)(L-, t) = 0$. This yields the self-adjoint differential equation

$$D^2(EID^2 u) - \omega^2 u = 0 \tag{3.10}$$

with linearly independent homogeneous boundary conditions (Kamke, [7, Chap. VIII]) $(D^2 u)(0) = (D^3 u)(0) = (D^2 u)(L) = (D^3 u)(L) = 0$. Via the substitution $\lambda^4 = (EI)^{-1}\omega^2$, equation (3.10) becomes

$$D^4 u - \lambda^4 u = 0 \tag{3.11}$$

with boundary conditions $(D^2 u)(0) = (D^3 u)(0) = (D^2 u)(L) = (D^3 u)(L) = 0$, which has solution

$$u(x) = \alpha_1 e^{\lambda x} + \alpha_2 e^{-\lambda x} + \alpha_3 \cos(\lambda x) + \alpha_4 \sin(\lambda x) , \tag{3.12}$$

where α_1, α_2, α_3 and α_4 satisfy the four homogeneous equations

$$\begin{bmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & \lambda^3 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ e^{\lambda L} & e^{-\lambda L} & -\cos(\lambda L) & -\sin(\lambda L) \\ e^{\lambda L} & -e^{-\lambda L} & \sin(\lambda L) & -\cos(\lambda L) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.13}$$

generated by the four linearly independent homogeneous boundary conditions. System (3.13) must possess a non-trivial solution for suitable λ's, thus either $\lambda = 0$ or the determinant of the second 4×4 matrix $A(\lambda)$ from the left must be zero – i.e.

$$\det(A(\lambda)) = 4[-1 + \cosh(\lambda L)\cos(\lambda L)] = 0 . \tag{3.14}$$

This necessitates either $\lambda = 0$ or $\cos(\lambda L) = \operatorname{sech}(\lambda L)$ ($\lambda \neq 0$), which after row reduction (for $\lambda \neq 0$) applied to the system (3.13) leads to

$$\begin{bmatrix} 1 & 0 & 0 & \left[-\frac{1}{2} + \frac{1}{2}\left\{\frac{\sinh(\lambda L) - \sin(\lambda L)}{\cosh(\lambda L) - \cos(\lambda L)}\right\}\right] \\ 0 & 1 & 0 & \left[\frac{1}{2} + \frac{1}{2}\left\{\frac{\sinh(\lambda L) - \sin(\lambda L)}{\cosh(\lambda L) - \cos(\lambda L)}\right\}\right] \\ 0 & 0 & 1 & \left[\frac{\sinh(\lambda L) - \sin(\lambda L)}{\cosh(\lambda L) - \cos(\lambda L)}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.15}$$

for $\lambda = \lambda_n$ (n a non-zero integer). These λ_n are the zeros of the meromorphic function $\operatorname{sech}(z) - \cos(z)$ of the complex variable ($z = x + iy$, $i \equiv \sqrt{-1}$), whose zeros λ_n are countably infinite and lie on the real axis with asymptotic behavior $\lim_{|n|\to\infty}[\lambda_n L - (n + 1/2)\pi] = 0$. The thus obtained λ's are used to solve for the coefficients α_i ($1 \leqslant i \leqslant 4$), in particular

$$\begin{aligned} \alpha_1(n) &= \cosh(\lambda_n L) - \cos(\lambda_n L) + \sin(\lambda_n L) - \sinh(\lambda_n L)\,, \\ \alpha_2(n) &= \cos(\lambda_n L) - \cosh(\lambda_n L) - \sinh(\lambda_n L) + \sin(\lambda_n L)\,, \\ \alpha_3(n) &= 2(\sin(\lambda_n L) - \sinh(\lambda_n L))\,, \\ \alpha_4(n) &= 2(\cosh(\lambda_n L) - \cos(\lambda_n L))\,. \end{aligned} \tag{3.16}$$

and consequently

$$u_n(x) \equiv \alpha_1(n)\,e^{\lambda_n x} + \alpha_2(n)\,e^{-\lambda_n x} + \alpha_3(n)\cos(\lambda_n x) + \alpha_4(n)\sin(\lambda_n x) \tag{3.17}$$

is an eigenfunction corresponding to the eigenvalue λ_n ($n \neq 0$) with geometric multiplicity 1 for $n \neq 0$. Results Theorem 2.1 [1, Chapter 7], and [6, Chapters 8 and 9] imply that

$$\phi_n(x) \equiv \|u_n\|^{-1} u_n(x) \quad (n \neq 0)\,, \tag{3.18}$$

where

$$\begin{aligned} \|u_n\|^2 \equiv \int_{x=0}^{L} |u_n(x)|^2\,dx = \frac{1}{\lambda_n}\Big\{ & \frac{1}{2}\,[e^{2\lambda_n L} - 1]\alpha_1(n)^2 + \frac{1}{2}\,[1 - e^{-2\lambda_n L}]\alpha_2(n)^2 \\ & + \frac{1}{4}\,[2L\lambda_n + \sin(2L\lambda_n)]\alpha_3(n)^2 + \frac{1}{4}\,[2L\lambda_n - \sin(2L\lambda_n)]\alpha_4(n)^2 \\ & + 2L\lambda_n\alpha_1(n)\alpha_2(n) + [e^{\lambda_n L}(\cos(\lambda_n L) + \sin(\lambda_n L)) - 1]\alpha_1(n)\alpha_3(n) \\ & + [e^{\lambda_n L}(\sin(\lambda_n L) - \cos(\lambda_n L)) + 1]\alpha_1(n)\alpha_4(n) \\ & + [e^{-\lambda_n L}(\sin(\lambda_n L) - \cos(\lambda_n L)) + 1]\alpha_2(n)\alpha_3(n) + [1 - e^{-\lambda_n L}(\cos(\lambda_n L) \\ & + \sin(\lambda_n L))]\alpha_2(n)\alpha_4(n) + \left[\frac{1 - \cos(2\lambda_n L)}{2}\right]\alpha_3(n)\alpha_4(n)\Big\} \quad (n \neq 0)\,, \end{aligned} \tag{3.19}$$

is an orthonormal system of functions corresponding to eigenvalues λ_n ($n \neq 0$).

For eigenvalue $\lambda_0 = 0$ (singularity of product matrix in equation (3.13)), the general solution $u_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ of the homogeneous ordinary differential equation (3.10) must satisfy $(D^2 u)(0) = (D^3 u)(0) = (D^2 u)(L) = (D^3 u)(L) = 0$, hence $\alpha_2 = \alpha_3 = 0$ and $u_0(x) = \alpha_0 + \alpha_1 x$. Thus $\{\phi_0(x) \equiv L^{-1/2}, \Phi_0(x) \equiv (3L^{-1})^{1/2} - (12L^{-3})^{1/2} x\}$ is an orthonormal basis of eigenfunctions corresponding to eigenvalue $\lambda = 0$ with geometric multiplicity 2. This combined with relation (3.18) implies that

$$\mathbb{B} = \{\ldots, \phi_{-n-1}(x), \phi_{-n}(x), \ldots, \phi_{-1}(x), \phi_0(x), \Phi_0(x), \phi_1(x), \ldots, \phi_n(x), \phi_{n+1}(x), \ldots\} \tag{3.20}$$

is an orthonormal basis of the Hilbert space $L_2[0, L]$.

Due to the nature of q as a linear contribution of the modified Dirac delta functional and its derivative (Equation 3.2), a Fourier series expansion of q in terms of the eigenfunctions of equation (3.10) leads to delicate questions concerning the convergence of the series. To circumvent this, Hilbert space methods for the Hilbert space $L_2[0, L]$ involving the eigenvalues and the corresponding eigenfunctions of the self-adjoint differential equation must be resolved too. The Fourier series expansion via the orthonormal basis (3.20) of the space $L_2[0, L]$ for the $L_2[0, L]$ function f is

$$f(x) = \langle f, \Phi_0 \rangle \Phi_0(x) + \sum_{n=-\infty}^{\infty} \langle f, \phi_n \rangle \phi_n(x),$$

where

$$\begin{aligned} \langle f, \Phi_0 \rangle &= \int_{x=0}^{L} f(x) \Phi_0(x)\, dx, \\ \langle f, \phi_n \rangle &= \int_{x=0}^{L} f(x) \phi_n(x) dx \quad (n \in \mathbb{Z}). \end{aligned} \tag{3.21}$$

This series expansion of f is uniformly convergent if f has continuous third derivative on $[0, L]$ and satisfies homogeneous boundary conditions; otherwise, the equality holds almost everywhere (except on a set of Lebesgue measure 0) if f is an $L_2[0, L]$ function only.

Using orthonormal basis $\mathbb{B}$ of equation (3.20) and the fact that functions $\{\lambda_n^{-2} \phi_n''(x) : n \in \mathbb{Z}$ and $n \neq 0\}$ form an orthonormal set of $L_2[0, L]$

$$\{\lambda_m^{-2} \phi_m'', \lambda_n^{-2} \phi_n''\rangle = \delta_{mn} \quad (m, n \neq 0) \tag{3.22}$$

the Fourier series expansions of $w(x, t)$, $p(x, t)$ and $M(x, t)$ are

$$w(x, t) = \langle w(\cdot, t), \Phi_0 \rangle \Phi_0(x) + \sum_{n=-\infty}^{\infty} \langle w(\cdot, t), \phi_n \rangle \phi_n(x), \tag{3.23}$$

$$p(x, t) = \langle p(\cdot, t), \Phi_0 \rangle \Phi_0(x) + \sum_{n=-\infty}^{\infty} \langle p(\cdot, t), \phi_n \rangle \phi_n(x), \tag{3.24}$$

and

$$M(x, t) = \sum_{n=-\infty}^{\infty}{}' \langle M(\cdot, t), \lambda_n^{-2} \phi_n'' \rangle \lambda_n^{-2} \phi_n''(x) \tag{3.25}$$

respectively, where for arbitrary $L_2[0, L]$ functions $F(x, t)$ and $f(x)$ in variable x

$$\langle F(\cdot, t), f\rangle = \int_{x=0}^{L} F(x, t)f(x)\,\mathrm{d}x \tag{3.26}$$

and $\Sigma_{n=-\infty}^{\infty}{}'$ stands for summation over all non-zero integers. Substituting Fourier expansions ((3.23), (3.24) and (3.25)) into the Reissner functional J defined by Equation (3.8) and utilizing orthonormality of $\mathbb{B}$ in the Hilbert space $L_2[0, L]$ and relation (3.22)) reduce condition (3.8) to

$$\begin{aligned}
\min\Big\{ J(w, p, M) = &-\sum_{n=-\infty}^{\infty}{}' \lambda_n^2 \langle M(\cdot, t), \lambda_n^{-2}\phi_n''\rangle\langle w(\cdot, t), \phi_n\rangle \\
&+ \Big[\langle p(\cdot, t), \Phi_0\rangle\langle w(\cdot, t), \Phi_0\rangle + \sum_{n=-\infty}^{\infty} \langle p(\cdot, t), \phi_n\rangle\langle w(\cdot, t), \phi_n\rangle\Big] \\
&+ \frac{1}{2EI}\Big[\sum_{n=-\infty}^{\infty}{}' |\langle M(\cdot, t), \lambda_n^{-2}\phi_n''\rangle|^2\Big] + \frac{1}{k}\Big(\frac{1}{2}\Big[|\langle p(\cdot, t), \Phi_0\rangle|^2 \\
&+ \sum_{n=-\infty}^{\infty} |\langle p(\cdot, t), \phi_n\rangle|^2\Big] + \langle p(\cdot, t), \Phi_0\rangle(C\langle p(\cdot, \cdot\cdot), \Phi_0\rangle)(t) \\
&+ \sum_{n=-\infty}^{\infty} \langle p(\cdot, t), \phi_n\rangle(C\langle p(\cdot, \cdot\cdot), \phi_n\rangle)(t)\Big) + F(0)\langle w(\cdot, t), \Phi_0\rangle\Phi_0(0) \\
&+ F(0)\sum_{n=-\infty}^{\infty} \langle w(\cdot, t), \phi_n\rangle\phi_n(0) - M(0)\langle w(\cdot, t), \Phi_0\rangle\Phi_0'(0) \\
&- M(0)\sum_{n=-\infty}^{\infty} \langle w(\cdot, t), \phi_n\rangle\phi_n'(0)\Big\}.
\end{aligned} \tag{3.27}$$

Variation with respect to the sets of Fourier coefficients $\{\langle w(\cdot, t), \Phi_0\rangle\ \langle w(\cdot, t), \phi_n\rangle\ (n \in \mathbb{Z}),\ \langle p(\cdot, t), \Phi_0\rangle\ \langle p(\cdot, t), \phi_n\rangle\ (n \in \mathbb{Z}),\ \langle M(\cdot, t), \lambda_n^{-2}\phi_n''\rangle\ (n \in \mathbb{Z},\ n \neq 0)\}$ ((3.23), (3.24) and (3.25)) of $w(x, t)$, $p(x, t)$ and $M(x, t)$ in expression (3.27) of the Reissner functional J yields the following three sets of equations:

$$\begin{aligned}
&\langle p(\cdot, t), \Phi_0\rangle + F(0)\Phi_0(0) - M(0)\Phi_0'(0) = 0\,, \\
&-\lambda_n^2\langle M(\cdot, t), \lambda_n^{-2}\phi_n''\rangle + \langle p(\cdot, t), \phi_n\rangle + F(0)\phi_n(0) - M(0)\phi_n'(0) = 0 \quad (n \in \mathbb{Z})\,,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
&\langle w(\cdot, t), \Phi_0\rangle + \frac{1}{k}\,[\langle p(\cdot\,, t), \Phi_0\rangle + (C\langle p(\cdot, \cdot\cdot), \Phi_0\rangle)(t)] = 0\,, \\
&\langle w(\cdot, t), \phi_n\rangle + \frac{1}{k}\,[\langle p(\cdot\,, t), \phi_n\rangle + (C\langle p(\cdot, \cdot\cdot), \phi_n\rangle)(t)] = 0 \quad (n \in \mathbb{Z})
\end{aligned} \tag{3.29}$$

and

$$-\lambda_n^2\langle w(\cdot, t), \phi_n\rangle + \frac{1}{EI}\,\langle M(\cdot, t), \lambda_n^{-2}\phi_n''\rangle = 0 \quad (n \in \mathbb{Z}, n \neq 0)\,, \tag{3.30}$$

where for arbitrary suitable $F(x, t)$ and $f(x)$

$$(C\langle F(\cdot, \cdot\cdot), f\rangle)(t) = \int_{\tau=0}^{t} c(t-\tau)\langle F(\cdot, \tau), f\rangle\,\mathrm{d}\tau = (c * \langle F(\cdot, \cdot\cdot), f\rangle)(t)\,. \tag{3.31}$$

Because the operator C appearing in Equation (3.29) is a convolution integral operator, the solutions for the Fourier coefficients of the functions $\Phi_0(x)$, $\phi_n(x)$ $(n \in \mathbb{Z})$ and $\lambda_n^{-2}\phi_n''(x)$ $(n \in \mathbb{Z},\ n \neq 0)$ are

$$\langle p(\cdot\,,t), \Phi_0\rangle = -[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]\,,$$

$$\langle w(\cdot\,,t), \Phi_0\rangle = \frac{1}{k}\,[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]([I + C]1)(t)\,, \tag{3.32}$$

$$\langle p(\cdot,t), \phi_n\rangle = -\frac{k}{EI\lambda_n^4 + k}\,[F(0)\phi_n(0) - M(0)\phi_n'(0)]\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1} 1\right)(t)$$
$$(n \in \mathbb{Z})\,,$$

$$w(\cdot,t), \phi_n\rangle = \frac{1}{EI\lambda_n^4 + k}\,[F(0)\phi_n(0) - M(0)\phi_n'(0)]$$
$$\times\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1}[I + C]1\right)(t) \quad (n \in \mathbb{Z})$$

and

$$\langle M(\cdot,t), \lambda_n^{-2}\phi_n''\rangle = \frac{EI\lambda_n^2}{EI\lambda_n^4 + k}\,[F(0)\phi_n(0) - M(0)\phi_n'(0)]\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1}[I + C]1\right)(t)$$
$$(n \in \mathbb{Z},\ n \neq 0)\,, \tag{3.33}$$

where $[\ ]^{-1}$ denotes the inverse of the operator inside the brackets $[\ ]$. Hence, $w(x,t)$, $p(x,t)$ and $M(x,t)$ have the Fourier expansions

$$w(x,t) = \frac{1}{k}\,[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]([I + C]1)(t)\Phi_0(x) + \sum_{n=-\infty}^{\infty}\frac{1}{EI\lambda_n^4 + k}$$
$$\times[F(0)\phi_n(0) - M(0)\phi_n'(0)]\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1}[I + C]1\right)(t)\phi_n(x)\,, \tag{3.34}$$

$$p(x,t) = -[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]\Phi_0(x) - \sum_{n=-\infty}^{\infty}\frac{k}{EI\lambda_n^4 + k}\,[F(0)\phi_n(0) - M(0)\phi_n'(0)]$$
$$\times\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1} 1\right)(t)\phi_n(x) \tag{3.35}$$

and

$$M(x,t) = \sum_{n=-\infty}^{\infty}\frac{EI}{EI\lambda_n^4 + k}\,[F(0)\phi_n(0) - M(0)\phi_n'(0)]$$
$$\times\left(\left[I + \frac{EI\lambda_n^4}{EI\lambda_n^4 + k}\,C\right]^{-1}[I + C]1\right)(t)\phi_n''(x)\,. \tag{3.36}$$

The operator C, appearing in the Fourier expansions of $w(x,t)$, $p(x,t)$ and $M(x,t)$ ((3.34), (3.35) and (3.36)), respectively, is a scalar multiple of the fractional exponential convolution operator $\Xi_\alpha(-\beta)$ with kernel

$$\xi_\alpha(-\beta, t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp(zt)[z^{1+\alpha} + \beta]^{-1}\, dz$$
$$\sim -t^\alpha \sum_{n=\infty}^{\infty} \frac{(-1)^n[\beta t^{1+\alpha}]^{-n}}{\Gamma([1+\alpha][1-n])}, \tag{3.37}$$

In the integral representation (3.37) the complex variable $z = x + iy$ ($i \equiv \sqrt{-1}$) is taken over the straight line path in the complex plane from $a - i\infty$ to $a + i\infty$ ($a > 0$), $\Gamma(z)$ is the gamma function and $\sim$ denotes asymptotic representation for large t. Thus $C = \kappa\Xi_\alpha(-\beta)$ with creep parameters, α, β and κ to be determined from creep tests (discussed in a subsequent section of this publication).

Since the operators $I + C$ as well as $[I + \mu C]^{-1}$ in relations (3.34), (3.35) and (3.36) act on the constant function 1, one needs to use the formula

$$[\Xi_\alpha(-\beta)1](t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp(zt)[z^{2+\alpha} + \beta z]^{-1}\, dz$$
$$\sim \frac{1}{\beta} - t^{1+\alpha}\sum_{n=2}^{\infty} \frac{(-1)^n[\beta t^{1+\alpha}]^{-n}}{\Gamma(1+[1+\alpha][1-n])} \tag{3.38}$$

with its accompanying asymptotic expansion for very large values of t. In consequence of relations $C = \kappa\Xi_\alpha(-\beta)$, (3.37) and (3.38), the Fourier representations of $w(x, t)$, $p(x, t)$ and $M(x, t)$ and their respective expansions (derived from the asymptotic expansions of the Fourier coefficients) for very large t, take the forms, accompanied by asymptotic expansions of the Fourier coefficients for t very large:

$$w(x, t) = \frac{1}{k}[F(0)\Phi_0(0) - M(0)\Phi_0'(0)](I + \kappa\Xi_\alpha(-\beta))(t)\Phi_0(x)$$
$$+ \sum_{n=-\infty}^{\infty} \frac{1}{EI\lambda_n^4 + k}[F(0)\phi_n(0) - M(0)\phi_n'(0)]$$
$$\times\left(I + \frac{k\kappa}{EI\lambda_n^4 + k}\Xi_\alpha(-\beta - [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])\right)(t)\phi_n(x)$$
$$\sim \frac{1}{k}[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]\left(1 + \frac{\kappa}{\beta} - \kappa t^{1+\alpha}\sum_{m=2}^{\infty}\frac{(-1)^m(\beta t^{1+\alpha})^{-m}}{\Gamma(1+[1+\alpha][1-m])}\right)\Phi_0(x)$$
$$+ \sum_{n=-\infty}^{\infty}\frac{1}{EI\lambda_n^4 + k}[F(0)\phi_n(0) - M(0)\phi_n'(0)]\left(1 + \frac{k\kappa}{\beta(EI\lambda_n^4 + k) + EI\lambda_n^4\kappa}\right.$$
$$\left. - \frac{k\kappa}{EI\lambda_n^4 + k}t^{1+\alpha}\sum_{m=2}^{\infty}\frac{(-1)^m((\beta + [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])t^{1+\alpha})^{-m}}{\Gamma(1+[1+\alpha][1-m])}\right)\phi_n(x), \tag{3.39}$$

$$p(x, t) = -[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]\Phi_0(x) - \sum_{n=-\infty}^{\infty}\frac{k}{EI\lambda_n^4 + k}[F(0)\phi_n(0) - M(0)\phi_n'(0)]$$
$$\times\left(I - \frac{EI\lambda_n^4\kappa}{EI\lambda_n^4 + k}\Xi_\alpha(-\beta - [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])\right)(t)\phi_n(x)$$

$$\sim -[F(0)\Phi_0(0) - M(0)\Phi_0'(0)]\Phi_0(x)$$

$$- \sum_{n=-\infty}^{\infty} \frac{k}{EI\lambda_n^4 + k} [F(0)\phi_n(0) - M(0)\phi_n'(0)]\Big(1 - \frac{EI\lambda_n^4\kappa}{\beta(EI\lambda_n^4 + k) + EI\lambda_n^4\kappa}$$

$$+ \frac{EI\lambda_n^4\kappa}{EI\lambda_n^4 + k} t^{1+\alpha} \sum_{m=2}^{\infty} \frac{(-1)^m((\beta + [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])t^{1+\alpha})^{-m}}{\Gamma(1 + [1+\alpha][1-m])}\Big)\phi_n(x) \tag{3.40}$$

and

$$M(x,t) = \sum_{n=-\infty}^{\infty} \frac{EI}{EI\lambda_n^4 + k} [F(0)\phi_n(0) - M(0)\phi_n'(0)]$$

$$\times \Big(1 + \frac{k\kappa}{EI\lambda_n^4 + k} \Xi_\alpha(-\beta - [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])\Big)(t)\phi_n''(x)$$

$$\sim \sum_{n=-\infty}^{\infty} \frac{EI}{EI\lambda_n^4 + k} [F(0)\phi_n(0) - M(0)\phi_n'(0)]\Big(1 + \frac{k\kappa}{\beta(EI\lambda_n^4 + k) + EI\lambda_n^4\kappa}$$

$$- \frac{k\kappa}{EI\lambda_n^4 + k} t^{1+\alpha} \sum_{m=2}^{\infty} \frac{(-1)^m((\beta + [EI\lambda_n^4\kappa]/[EI\lambda_n^4 + k])t^{1+\alpha})^{-m}}{\Gamma(1 + [1+\alpha][1-m])}\Big)\phi_n''(x). \tag{3.41}$$

4. Direct determination of the parameters of the fractional exponential creep kernel from a pile test

The pile head displacement $e(t)$ as function of time has representation in terms of the fractional exponential function

$$e(t) = e_0(1 + \kappa \Xi_\alpha(-\beta)1) \tag{4.1}$$

with parameters e_0, κ, α and β to be determined from creep tests (Rabotnov [16]). One ascertains the four parameters e_0, κ, α and β from an experimental curve after Laplace transforming (Doetsch [2]) expression (4.1)

$$\Psi(s) = e_0\Big(1 + \frac{\kappa}{s^{1+\alpha} + \beta}\Big). \tag{4.2}$$

Following a procedure of [16, Chap. III], one calculates the function $\Psi(s)$ in the form

$$\Psi(s) = s \int_{t=0}^{\infty} e(t)\, \mathrm{e}^{-st}\, \mathrm{d}t \tag{4.3}$$

and determines optimum approximation-parameters e_0, κ, α and β via least square minimization (of a suitable evaluation function, defined later).

This procedure is applied to the pile test results obtained at Dawson City (1973) [9]. Initially, one tabulates $e(t)$-values $e_i = e(t_i)$ for certain t-values t_i $(1 \le i \le n)$ for pile-test results (Table 1(a)), where $t_n = T$ is the largest t for which the value $e(t)$ is known. Then integral (4.3) is rewritten as

$$\Psi(s_k) = s_k\Big\{\sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} e(t)\, \mathrm{e}^{-s_k t}\, \mathrm{d}t + \int_{t=T}^{\infty} e(t)\, \mathrm{e}^{-s_k t}\, \mathrm{d}t\Big\} \quad (1 \le k \le p), \tag{4.4}$$

Table 1. Tabulation of soil parameters

Dawson City S-2-L pile					
(a) Test results		(b) Laplace transformed results		(c) Optimal soil parameters	
$t(h)$	$e(t)$ (mm)	s_r	$\Psi(s_r)$		
0	2.70	0.05	5.6263	α	-0.175
2	4.17	0.06	5.5125	β	$0.274\,(\mathrm{hr})^{0.175}$
4	4.72	0.07	5.4117	e_0	2.590 mm
6	5.29	0.08	5.3208	κ	$0.419\,(\mathrm{hr})^{0.175}$
8	5.48	0.09	5.2378		
10	5.86	0.1	5.1611		
12	5.96	0.2	4.6124		
15	6.10	0.3	4.2792		
20	6.13	0.4	4.0525		
30	6.23	0.5	3.8867		
40	6.32	0.6	3.7593		
50	6.41	0.7	3.6576		
60	6.52	0.8	3.5743		
80	6.69	0.9	3.5045		
100	6.87	1.0	3.4451		
120	7.10	1.1	3.3939		
140	7.24	1.2	3.3492		
160	7.24	1.3	3.3099		
180	7.61	1.4	3.2750		

where the last integral becomes negligible for large t (assuming $e(t)$ as Laplace-transformable). Hence, for tabulation purposes

$$\Psi(s_k) = s_k \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} e(t)\, e^{-s_k t}\, \mathrm{d}t \quad (1 \leq k \leq p)\,. \tag{4.5}$$

For the evaluation of the integral (4.3) by means of (4.5), $e(t)$ is assumed to have the quadratic polynomial form

$$e(t) = d_{0i} + d_{1i}t + d_{2i}t^2 \quad (t_{i-1} < t \leq t_i,\ 1 \leq i \leq n) \tag{4.6}$$

on each partition interval. The quadratic interpolation polynomial $d_{0i} + d_{1i}t + d_{2i}t^2$ is through the three successive points (t_{i-1}, e_{i-1}), (t_i, e_i) and (t_{i+1}, e_{i+1}) and the polynomial expression for $e(t)$ holds on the interval from $[t_{i-1}, t_{i+1}]$ beginning with $t_0 = 0$. These interpolation polynomials are thereafter used to evaluate the integrals in equations (4.5) which gave the $\Psi(s_k)$-values (Table 1(b)). Finally, the optimal parameter-values for e_0, κ, α and β are determined via minimization of the evaluation function

$$\gamma(\kappa, \alpha, \beta, e_0) = \sum_{k=1}^{p} \left(\Psi(s_k) - e_0 - \frac{e_0 \kappa}{s_k^{1+\alpha} + \beta} \right)^2 . \tag{4.7}$$

This optimization yielded the soil parameters κ, α, β and $k = F/(e_0 L)$ (Table 1(c)) used herein for the long range prediction ($t \to \infty$) of the pile behavior of the S-2-L pile (Dawson City, 1973). The solid curve in Fig. 2 is the plotting of $e(t)$ by means of formula (4.1) (for these determined soil parameters e_0, κ, α and β) and agrees closely with the test results.

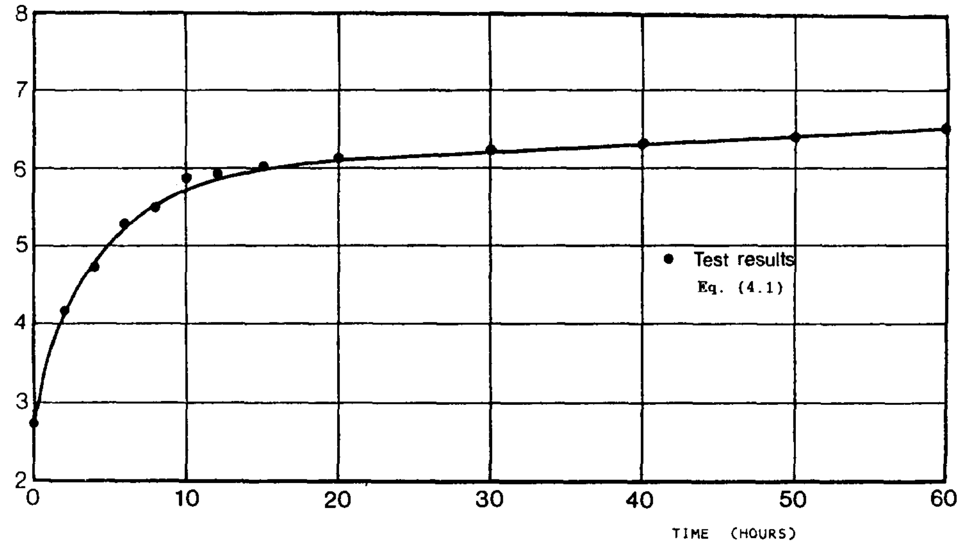

Fig. 2. Comparison of fractional exponential function, Eq. (4.1), with the measured pile displacements. Pile S-2-L, Dawson City.

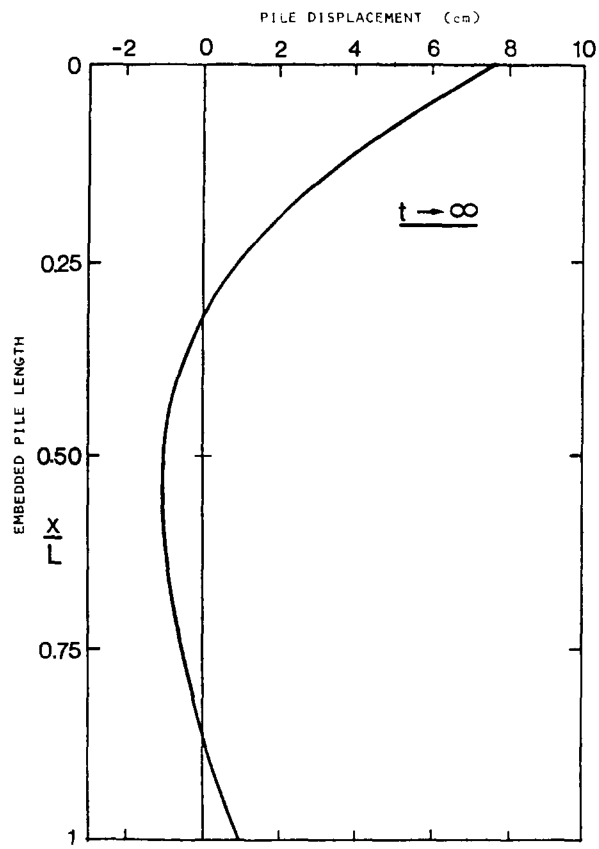

Fig. 3. Asymptotic estimate, by Eq. (3.39), of pile displacements versus embedded pile length. Pile S-2-L, Dawson City.

5. Protracted pile behavior from field data

The herein derived expressions for displacement, pressure and moments for a laterally loaded pile in permafrost are utilized to forecast the long term behavior ($t \to \infty$) from the available field data (Dawson City, 1973). At Dawson City the two methods of pile installations, namely the conventional "annual backfill" and the underdrill-drive technique were used. In the latter technique, a triple-wall reverse air flow system removes cuttings from a 44.45 cm bit attached to the pile tip, while the bit and pile advance by a hammer delivering 24.42 kN · m per blow through the two inner pipe strings. The 45.72 cm pile is forced into the 44.45 cm diameter hole. Both installation techniques for the test piles involved considerable lateral soil displacements. Probably soil-prestrain during pile installation produced changes in soil-properties, which in turn affect pile-behavior and account for some of the discrepancies between field data and forecast.

The asymptotic form of the solution ((3.39), (3.40) and (3.41)) was used to predict pile behavior from the cited experimental results. However, extrapolation beyond maximum testing time is somewhat delicate, because ground temperature variations and frozen soil consolidation occur over long time spans. These two factors induce significant deviations of predicted behavior from relatively short-term observations.

Finally, asymptotically predicted displacements (3.39), reactions (3.40) and moments (3.41) along the pile length are illustrated in Figs 3, 4 and 5 and the respective asymptotic quantities $t \to \infty$ are calculated.

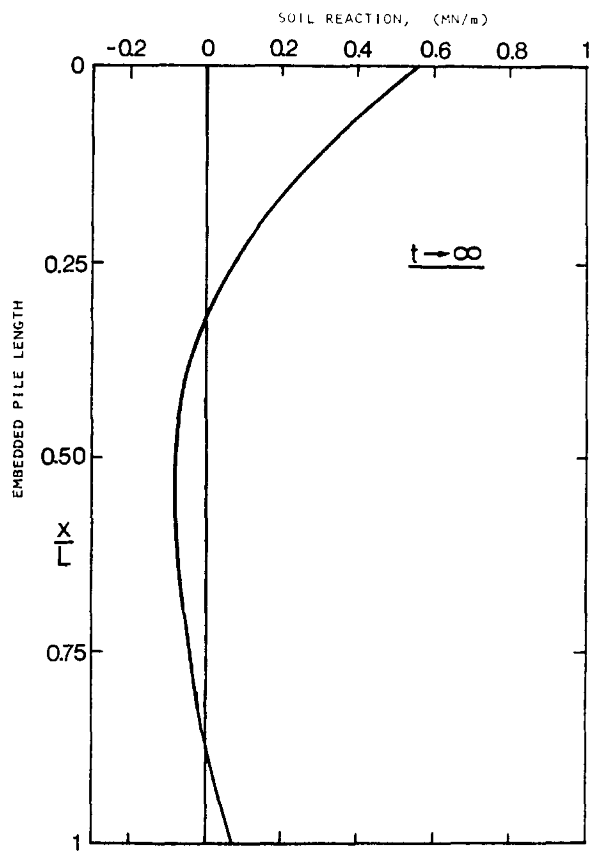

Fig. 4. Asymptotic estimate, by Eq. (3.40), of soil reaction versus embedded pile length. Pile S-2-L, Dawson City.

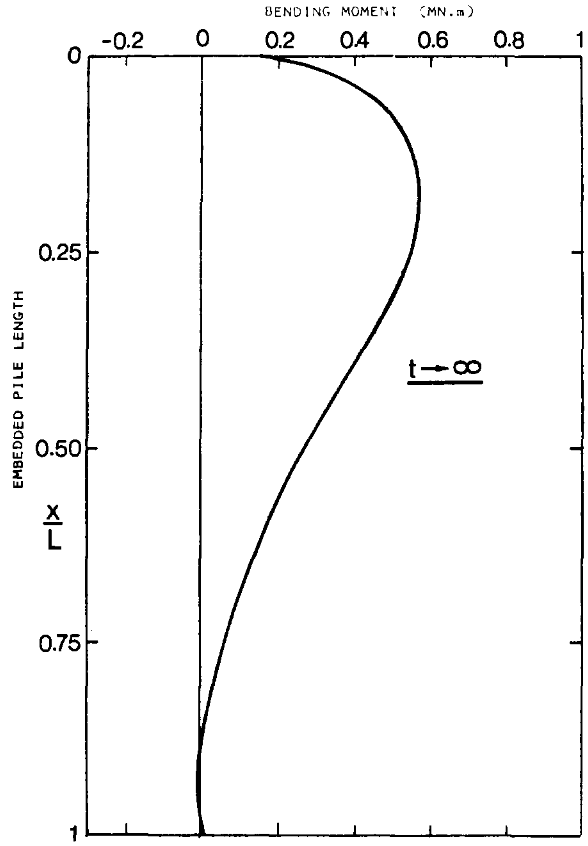

Fig. 5. Asymptotic estimate, by Eq. (3.41), of bending moment versus embedded pile length. Pile S-2-L, Dawson City.

6. Conclusion

The calculus of variation is an excellent tool for predicting laterally loaded pile behavior in permafrost. Its computational difficulties are significantly less than those of other methods of analysis – e.g. finite difference and finite element methods. Moreover, the solutions are continuous on the entire domain, as opposed to mere piecewise continuity of the finite difference and finite element methods. Specifically, the mixed formulation via the Reissner Functional J in equation (3.8) guarantees the continuity of the primary variables of displacement, reaction and moment. This is generally false for the usual standard formulations, which assure continuity of displacement only, but not that of moment and reaction.

Conclusions drawn from asymptotic expansions in variable t are only valid for very large t values because of the nature of asymptotic convergence – i.e. stationary creep.

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